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Stable splittings of BP for some P of order thirty-two

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Abstract

In this paper, we use the method of Benson and Feshbach outlined in "Stable splittings of classifying spaces of finite groups" (in Topology, Vol. 31, No. 1 (1992)) to give the complete, 2-complete stable splitting of the classifying space BP for all but two of the groups P of order thirty-two which are not direct products in a nontrivial way. One of these two is $\mathbf{Z}/32$, and $B(\mathbf{Z}/32)$ is known to be indecomposable. For the other, P is the semidirect product of $(\mathbf{Z}/2)^4$ and $\mathbf{Z}/2$. This is the only nonabelian group of order 32 which is not a direct product and has a subgroup isomorphic to $(\mathbf{Z}/2)^4$. We also give some results which refine the method of Benson and Feshbach. We have found that BP is indecomposable for three of these groups, and these are the first examples of indecomposable BP for which P is a nonabelian 2-group. Poincaré series are given for each classifying space using Rusin's paper "The cohomology of the groups of order 32" (Mathematics of Computation, Vol. 53, No. 187 (1989)).

0. Introduction

In this paper, we give the complete, 2-complete stable splitting of the classifying space BP for all but two of the groups P of order thirty-two which are not direct products in a nontrivial way. One of these two is $\mathbf{Z}/32$, and $B(\mathbf{Z}/32)$ is known to be indecomposable. For the other, P is the semidirect product of $(\mathbf{Z}/2)^4$ and $\mathbf{Z}/2$. This is the only nonabelian group of order 32 which is not a direct product and has a subgroup isomorphic to $(\mathbf{Z}/2)^4$. There are 32 such groups P. The main reason we chose these groups to consider was to find indecomposable BP for which P was not cyclic. It turns out that BP is indecomposable for three of these groups, and these are the first examples of indecomposable BP for which P is a nonabelian 2-group. This is interesting in light of Theorems 1.7 and 1.8 of [9], which together imply that, for all

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$n \geq 2$, 'almost all' p-groups P of *Frattini length* n have indecomposable classifying spaces.

Stable splittings have been provided for some groups P of order 32, although not all of these have been complete splittings. These include the abelian groups (see [8]), the extra-special groups (see [4, 5]), the dihedral group D_{32}, the quaternion group Q_{32} (see [13]), $\mathbf{Z}/4 \wr \mathbf{Z}/2$, the semihedral group SD_{32}, and the quasidihedral group (see [11]). We give complete splittings for all of these except the abelian groups.

We use the method of Benson and Feshbach outlined in their paper 'Stable splittings of classifying spaces of finite groups' (see [1]). The foundation for this method is a generalization of Segal's conjecture proved by Lewis et al. [10], which asserts that the p-completion of the double Burnside ring $A(P,P)$ is isomorphic to the ring of p-complete stable maps $\{BP, BP\}$. A consequence of this is that there is a one-to-one correspondence between the simple $\bar{A}(P,P)$-modules and the indecomposable stable summands of BP, where $\bar{A}(P,P) = \mathbf{Z}/p \otimes A(P,P)$. Benson and Feshbach show how to find stable splittings by giving a method for constructing the simple $\bar{A}(P,P)$-modules. As opposed to most other methods of finding stable splittings, this method makes no explicit use of cohomology.

Although we are mostly interested in what happens when $p = 2$, the general theory outlined in [1] holds for any prime p, and so throughout this paper, P is an arbitrary p-group, BP stands for the p-completion of the suspension spectrum of the classifying space with disjoint basepoint attached, and $\{BP, BP\}$ the ring of (p-complete) maps from BP to itself. In particular, $B1$ is the sphere spectrum S^0, and appears as a stable summand of BP for all P. We let $\overline{BP} = BP/B1$, so that $BP \simeq B1 \vee \overline{BP}$. Unless otherwise specified, all maps and spectra are in the homotopy category of p-complete spectra. We let ${}^xK$ denote the conjugate xKx^{-1} of a group K, and S will always be a simple $\mathbf{F}_p\mathrm{Out}K$-module.

Section 1 is a short section of preliminaries. In Section 2, we outline the method of [1]. Section 3 contains some results from [2] which refine the method of [1]. Two examples of complete splittings are outlined in Section 4, including one case where BP is indecomposable. Section 5 includes tables which give the stable splitting of BP for each P as described above, and the Poincaré series for the indecomposable spectra which appear as summands of the spectra BP. For the most part, spectra will be labelled by their *vertex* and *source* (see [1], Section 5, or Section 2 below). The Poincaré series are arrived at using Rusin's paper 'The cohomology of the groups of order 32' (see [15]), which gives the Poincaré series for all the groups of order 32.

1. Preliminaries

As is shown, for example, in Nishida [14, Section 4], a complete stable splitting $BP \simeq X_1 \vee \cdots \vee X_n$ corresponds to a primitive decomposition $1 = e_1 + \cdots + e_n$ of the identity in $\{BP, BP\}$, where the e_i are primitive, mutually orthogonal idempotents in $\{BP, BP\}$. Given a splitting, an idempotent decomposition is given by letting $e_i = i_i \circ \pi_i$, where i_i is the inclusion of X_i into BP and π_i is the projection from BP onto X_i. We call e_i the idempotent splitting X_i from BP, and write $X_i = e_i BP$. We have $X_i \simeq X_j$ iff e_i is conjugate to e_j iff $e_i\{BP, BP\} \cong e_j\{BP, BP\}$. As is shown by Lewis et al. in [10], $\{BP, BP\} \cong \widehat{A}(P,P) = \hat{\mathbf{Z}}_p \otimes A(P,P)$, and so stable splittings of BP also correspond to primitive idempotent decompositions of 1 in $\widehat{A}(P,P)$. Reducing $\hat{\mathbf{Z}}_p$ to its residue field $\mathbf{Z}/p$, we get maps

$$\{BP, BP\} \xrightarrow{\cong} \widehat{A}(P,P) \twoheadrightarrow \bar{A}(P,P) = \mathbf{Z}/p \otimes A(P,P)$$

The idempotent refinement theorem then implies that an idempotent decomposition of 1 in $\{BP, BP\}$ can be obtained by lifting a decomposition in $\bar{A}(P,P)$. We get a one-to-one correpondence between stable homotopy types of summands of BP and isomorphism types of simple $\bar{A}(P,P)$-modules. Letting $\bar{e}_i$ denote the image in $\bar{A}(P,P)$ of an idempotent e_i in $\{BP, BP\}$, we have the multiplicity of e_iBP as a wedge summand of BP equals the dimension of the simple $\bar{A}(P,P)$-module corresponding to $\bar{e}_i$ over its endomorphism ring. As the endomorphism ring is a finite division ring, it is also a field. It follows that finding all the simple $\bar{A}(P,P)$-modules and computing their dimensions over their endomorphism rings will give us a complete splitting of BP. A method for constructing all the simple $\bar{A}(P,P)$-modules, given P, is outlined by Benson and Feshbach in [1].

2. The method of Benson and Feshbach

In this section, we outline the notation and main results from [1] which we use in finding our splittings.

In [1], the authors construct $\bar{A}(P,P)$-modules $\bar{L}(P,K)$ and $\bar{L}(P,K,S)$ for each *type* of subgroup K of P and each simple $\mathbf{F}_p\mathrm{Out}K$-module S. The definition of *type* is somewhat technical (see 4.3 of [1]), but in practice, type gives an only very slightly finer partition of the subgroups of P than isomorphism does. The $\bar{L}(P,K,S)$ are either simple or zero (see 5.7 of [1]), and there are conditions for $\bar{L}(P,K,S)$ to be nonzero (see 5.2 of [1], or 2.3 below).

As in [1], we denote the generators of $\bar{A}(P,P)$ as $\zeta_{H,\phi}$, where $H \leq P$ and ϕ is a homomorphism from H to P, and the generators of $\bar{L}(P,K)$ as $\bar{f}_{K',\psi}$, where $K, K' \leq P$ and ψ is an isomorphism from K' to K. The $\bar{L}(P,K)$ are also $\mathbf{F}_p\mathrm{Out}K$-modules via the action $\eta \cdot \bar{f}_{K',\psi} = \bar{f}_{K',\eta\circ\psi}$. The $\bar{L}(P,K,S)$ are defined as $S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K)/\mathscr{M}$, where S is a simple right $\mathbf{F}_p\mathrm{Out}K$-module and $\mathscr{M}$ is defined in Definition 2.2. Lemma 2.1 gives the structure of $S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K)$ as an $\bar{A}(P,P)$-module (see [1, Proposition 4.8]):

Lemma 2.1. *As $\mathbf{F}_p$-spaces, we have $S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K) \cong \bigoplus_{i=1}^n S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{f}_{K_i,\psi_i}$, where n is the number of conjugacy classes of subgroups of P of the same type as K, K_i*

is a conjugacy class representative, and $\psi_i : K_i \xrightarrow{\cong} K$. *The action of* $\bar{A}(P,P)$ *on* $S \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)$ *is given by*

$$\zeta_{H,\phi} \cdot (s \otimes \bar{f}_{K',\psi}) = \sum_{\substack{x \in [K' \backslash P/H] \\ K' \leq \mathrm{Stab}_P(S) \leq {}^xH \\ K' \cap {}^x(\ker \phi) = 1}} s \otimes \bar{f}_{\phi({}^{x^{-1}}K'),\psi \circ c_x \circ \phi^{-1}}. \tag{2.1}$$

The sum here runs over double coset representatives x of K' and H in P. The two conditions insure that $\psi \circ c_x \circ \phi^{-1}$ is a well-defined homomorphism from the subgroup $\phi({}^{x^{-1}}K')$ of P to P. The condition $K' \leq {}^xH$ insures that ϕ is defined on ${}^{x^{-1}}K'$. The ambiguity created by ϕ^{-1} is in ${}^{x^{-1}}K' \cap \ker \phi$, which is the trivial group under the second condition in the sum. Note that any double coset $K'xH$ contains the coset xH. If $K' \leq {}^xH$, then ${}^xH = K'{}^xH = K'xHx^{-1}$, which implies that $xH = K'xH$. So, we may take the sum in 2.1 to be over coset representatives $x \in [P/H]$ satisfying the two conditions. $\mathrm{Stab}_P(S)$ is the kernel of the composition $N_P(K') \to \mathrm{Out}(K') \cong \mathrm{Out}K \to \mathrm{Aut}S$, where the first map is induced by conjugation and the last by the right action of $\mathrm{Out}K$ on S. Note that the kernel of the first map is $K'C_P(K')$ and so $K'C_P(K') \leq \mathrm{Stab}_P(S)$, and that $\mathrm{Stab}_P(S)$ depends on the subgroup K' of P. In [1], following 4.8, the authors remark that formula (2.1) also holds if we ignore the condition $\mathrm{Stab}_P(S) \leq {}^xH$, and only assume $K' \leq {}^xH$.

The submodule $\mathscr{M}$ in the definition of $\bar{L}(P,K,S)$ is defined as follows ([1, Definition 5.8]):

Definition 2.2. Let $\mathscr{M}_0$ equal $\bigcap_{\mathrm{im}\,\phi \cong K} \ker(\zeta_{L,\phi})$ which is an $\bar{A}(P,P)$ submodule of $S \otimes_{\mathbf{F}_p \mathrm{Out}K} \bar{L}(P,K)$. Inductively, define $\mathscr{M}_i$ to be the preimage in $S \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)$ of the submodule $\bigcap_{\mathrm{im}\,\phi \cong K} \ker(\zeta_{L,\phi})$ of $S \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)/\mathscr{M}_{i-1}$. As $\bar{A}(P,P)$ is finite, we eventually have $\mathscr{M}_i = \mathscr{M}_{i-1}$ and we let $\mathscr{M} = \mathscr{M}_i$ for any such i.

We remarked in [2] that, in the intersections in 2.2, we may assume $\mathrm{Stab}_P(S) \leq L$, since otherwise $\ker(\zeta_{L,\phi}) = S \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)$, by (2.1).

If $\bar{L}(P,K,S)$ is simple, we let $X_{K,S}$ denote the indecomposable summand of BP corresponding to $\bar{L}(P,K,S)$. We call K the *vertex* and S the *source* of the summand $X_{K,S}$. The vertex is uniquely determined up to the type of subgroup of P, and S is uniquely determined up to isomorphism by the summand $X_{K,S}$ (see 5.7 and 5.8 of [1]).

The following result, which is similar to Corollary 4.1 of Martino and Priddy [12], gives equivalent conditions for which $\bar{L}(P,K,S)$ is simple. The only real difference between Theorem 2.3 and Theorem 5.2 of [1] is the additional condition $\mathrm{Stab}_P(S) \leq H$. As in [1, 14], we say that a summand X of BK is *dominant* if it is not equivalent to a summand of BK' for $K' < K$. We will let * denote the duality automorphism of $\mathbf{F}_p\mathrm{Out}K$ given by $g \mapsto g^{-1}$ and S^* the left $\mathbf{F}_p\mathrm{Out}K$-module dual to S.

Theorem 2.3. *Let X be a dominant summand of BK with corresponding idempotent $\bar{e}_0 \in \mathbf{F}_P\,\mathrm{Out}K$ and S the simple right $\mathbf{F}_p\mathrm{Out}K$-module such that $\bar{e}_0 S \neq 0$. The following are equivalent*:

(i) *X is equivalent to a summand eBP of BP.*

(ii) *K is isomorphic to a subgroup $K' \leq P$ with the following properties. There is a split surjection $\pi : H \to K$ for some $H \leq P$, where $\mathrm{Stab}_P(S) \leq H \leq P$, and the element $\gamma \in \mathbf{F}_p\mathrm{Out}K$ given by*

$$\gamma = \sum_{\substack{x\in[K'\backslash P/H]\\ K'\leq^x H}} K \cong K' \xrightarrow{c_{x^{-1}}} {}^{x^{-1}}K' \hookrightarrow H \overset{\pi}{\twoheadrightarrow} K$$

satisfies $\gamma S^* \neq 0$. Furthermore, the idempotent e splitting X from BP may be taken to factor through $Bi : BK \to BP$.

If condition (i) of Theorem 2.3 is satisfied, then $X \simeq X_{K,S}$.

In practice, we will let $K = K'$. Also, note that condition (ii) is satisfied with a particular K' iff it is satisfied with ${}^{x^{-1}}K'$, for any $x \in P$, so we may assume that $K' \leq H$ and that the composition $K' \hookrightarrow H \overset{\pi}{\twoheadrightarrow} K'$ is the identity. So, for instance, if $H = P$, we get $\gamma = \mathrm{id}_{K'}$ and K' is obviously a vertex in this case. If γ as in Theorem 2.3 satisfies $\gamma S^* \neq 0$ for a particular $K = K' \leq P$, we say that K *contributes* to the multiplicity of X in BP, and call K a *contributor* or say that K contributes to BP. We have that a group G is a vertex of BP if and only if G is isomorphic to a subgroup K of P which is a contributor. Note that all subgroups of P isomorphic to G are referred to as vertices if any one of them is a contributor. However, not all need be contributors.

Compare the expression for γ in Theorem 2.3 with formula (2.1). Note that the composition $K \cong K' \xrightarrow{c_{x^{-1}}} {}^{x^{-1}}K' \hookrightarrow H \overset{\pi}{\twoheadrightarrow} K$ is an automorphism if and only if ${}^{x^{-1}}K' \cap \ker\pi = 1$. So, the sum in the expression for γ is over the same two conditions in the sum of (2.1). Thus, γ will have the same number of nonzero summands as appear in the formula for $\zeta_{H,\phi}(s \otimes \bar{f}_{K',\psi})$ for any isomorphism $\psi : K' \to K$.

Lemma 2.1, Definition 2.2 and Theorem 2.3 are the main components of the method of [1]. For convenience, we also state two results which are closely derived from 5.9 and 5.10 of [1], and which will be referred to in the next section.

Proposition 2.4. *Let X be a summand of BP with vertex K and source S. Let $K' \leq P$ with $K' \cong K$. If K' contributes to the multiplicity of X in BP, then K' is a direct summand of both $\mathrm{Stab}_P(S)$ and $K'C_P(K')$.*

The spectrum $X_{K,\mathbf{F}_p}$ is called the *principal dominant summand* of BK.

Corollary 2.5. *Let $\mathbf{F}_p$ denote the trivial $\mathbf{F}_p\mathrm{Out}K$-module. Let $K' \leq P$ with $K' \cong K$. If K' contributes to the multiplicity of the principal dominant summand of BK in BP,* *then K' is a direct summand of $N_P(K')$. In particular, if* Out$K$ *is a p-group, then K' being a contributor to BP implies that K' is a direct summand of $N_P(K')$.*

This follows directly from Proposition 2.4 and the fact that $\mathrm{Stab}_P(\mathbf{F}_p) = N_P(K')$.

3. Refinements of the method

The following two lemmas from [2] greatly simplify the implementation of the method of [1]. The first facilitates the calculation of $\bar{L}(P,K,S)$. The second simplifies the calculation of the endomorphism ring of $\bar{L}(P,K,S)$. It is similar in flavor to Proposition 1.4 of [12].

Lemma 3.1. *In Definition 2.2, $\mathcal{M} = \mathcal{M}_0$*

Lemma 3.2. *If $\bar{L}(P,K,S) \neq 0$, then*

$$\mathrm{End}_{\bar{A}(P,P)}(\bar{L}(P,K,S)) \cong \mathrm{End}_{\mathbf{F}_p \mathrm{Out}\, K}(S)$$

as vector spaces over $\mathbf{F}_p$.

Before proving these lemmas, we repeat some remarks concerning formula (2.1) which are made in [2]. *In all of the following remarks, x is always a double coset representative of K' and H in P satisfying the two conditions in* (2.1). *We let K, $K', H \leq P$ with $K' \cong K$, and we assume all $\bar{f}_{K',\psi}$ are in $\bar{L}(P,K)$ and that $\phi : H \twoheadrightarrow K_0 \cong K$.*

Remark 3.3. Since the subgroup $\phi(H) = K_0$ of P is isomorphic to K', we must have that $\phi({}^{x^{-1}}K') = \phi(H)$ for all x, and so $s \otimes \bar{f}_{\phi({}^{x^{-1}}K'),\psi \circ c_x \circ \phi^{-1}} = s \otimes \bar{f}_{K_0,\psi \circ c_x \circ \phi^{-1}}$. Now, $\psi \circ c_x \circ \phi^{-1}$ is an isomorphism from $K_0 = \phi({}^{x^{-1}}K')$ to K. We may write $\psi \circ c_x \circ \phi^{-1} = \eta_x \circ \psi_0$ where $\psi_0 : K_0 \xrightarrow{\cong} K$ and η_x is an automorphism of K which depends on x. We have

$$\begin{aligned} \zeta_{H,\phi} \cdot (s \otimes \bar{f}_{K',\psi}) &= \sum_{\substack{x \in [K' \backslash P / H] \\ K' \leq \mathrm{Stab}_P(S) \leq {}^x H \\ K' \cap {}^x(\ker \phi) = 1}} s \otimes \bar{f}_{\phi({}^{x^{-1}}K'),\psi \circ c_x \circ \phi^{-1}} \\ &= \sum s \otimes \bar{f}_{K_0,\eta_x \circ \psi_0} \\ &= \left(\sum s\eta_x\right) \otimes \bar{f}_{K_0,\psi_0}. \end{aligned}$$

Now, this can be done for any $s \otimes \bar{f}_{K',\psi}$. So, if we let $m = \sum_{i=1}^n s_i \otimes \bar{f}_{K_i,\psi_i}$ with $K_i \cong K'$ for all i, and let s_i' be the element $(\sum s_i \eta_x)$, we get

$$\begin{aligned}\zeta_{H,\phi}\cdot m &= \sum_{i=1}^{n}\zeta_{H,\phi}\cdot(s_i\otimes \bar{f}_{K_i,\psi_i})\\ &= \sum_{i=1}^{n} s_i'\otimes \bar{f}_{K_0,\psi_0}\quad \text{(from above)}\\ &= \left(\sum_{i=1}^{n} s_i'\right)\otimes \bar{f}_{K_0,\psi_0}. \end{aligned}\tag{3.1}$$

When applying (3.1), we denote the element $\sum s_i'$ by s_0 and write $\zeta_{H,\phi}\cdot m = s_0\otimes \bar{f}_{K_0,\psi_0}$. Note that $\zeta_{H,\phi}\cdot m$ may also equal $t\otimes \bar{f}_{K_0,\psi_0}$ for some $t\neq s_0$. However, when we write $\zeta_{H,\phi}\cdot m = s_0\otimes \bar{f}_{K_0,\psi_0}$, we always mean for s_0 to be equal to the element given by (3.1).

Remark 3.4. Let $a\in \mathbf{F}_p\mathrm{Out}K_0$. We may write $a=\sum_{i=1}^{k}\eta_i$ where the η_i are (not necessarily distinct) automorphisms of K_0. By $\zeta_{H,a\circ\phi}$ we mean $\sum_{i=1}^{k}\zeta_{H,\eta_i\circ\phi}$. Similarly, if $a'=\sum_{j=1}^{N}\eta_j'\in \mathbf{F}_p\mathrm{Out}K'$, we let $\bar{f}_{K',\psi\circ a'}=\sum_{j=1}^{N}\bar{f}_{K',\psi\circ\eta_j'}$.

Let $\psi_1 : K_1 \xrightarrow{\cong} K$ and let $\tilde{\psi} : K_0 \xrightarrow{\cong} K_1$ so that $\psi_0=\psi_1\circ\tilde{\psi}$. Let $m = s\otimes \bar{f}_{K',\psi}$, and let $\zeta_{H,\phi}\cdot m = s_0\otimes \bar{f}_{K_0,\psi_0}$ as in (3.1). Note that $\tilde{\psi}\circ\phi$ is a homomorphism from H to K_1. We get

$$\begin{aligned}\zeta_{H,\tilde{\psi}\circ\phi}\cdot m &= \sum_{\substack{x\in[K'\backslash P/H]\\ K'\le \mathrm{Stab}_P(S)\le {}^xH\\ K'\cap{}^x(\ker\tilde{\psi}\circ\phi)=1}} s\otimes \bar{f}_{(\tilde{\psi}\circ\phi)(x^{-1}K'),\psi\circ c_x\circ\phi^{-1}\circ\tilde{\psi}^{-1}}\\ &= \sum s\otimes \bar{f}_{K_1,\eta_x\circ\psi_1}\\ &= \left(\sum s\eta_x\right)\otimes \bar{f}_{K_1,\psi_1}\\ &= s_0\otimes \bar{f}_{K_1,\psi_1}.\end{aligned}$$

The same holds for any $m\in S\otimes_{\mathbf{F}_p\mathrm{Out}K}\bar{L}(P,K)$. The upshot is that if $\zeta_{H,\phi}\cdot m = s_0\otimes \bar{f}_{K_0,\psi_0}$, there is a map $\phi' : H \twoheadrightarrow K_1$ so that $\zeta_{H,\phi'}\cdot m = s_0\otimes \bar{f}_{K_1,\psi_1}$, where we can pick K_1 to be any subgroup of P isomorphic to K and ψ_1 any isomorphism form K_1 to K. Now, let s_1 be any element of S and assume that the s_0 above is not zero. Since S is simple, there is $a_0\in \mathbf{F}_p\mathrm{Out}K$ so that $s_1 = s_0a_0$. We can write $a_0\circ\psi_1=\psi_1\circ a^*$ for some $a\in \mathbf{F}_p\mathrm{Out}K_1$. Repeating the above argument with $\tilde{\psi}$ replaced by $a\circ\tilde{\psi}$, we get

$$\zeta_{H,a\circ\tilde{\psi}\circ\phi}\cdot m = s_0\otimes \bar{f}_{K_1,\psi_1\circ a^*} = s_0\otimes \bar{f}_{K_1,a_0\circ\psi_1} = s_1\otimes \bar{f}_{K_1,\psi_1}.$$

So, if $s_0\neq 0$, given any element of the form $s_1\otimes \bar{f}_{K_1,\psi_1}$, there is an element $\zeta\in \bar{A}(P,P)$ so that $\zeta\cdot m = s_1\otimes \bar{f}_{K_1,\psi_1}$, and we can take ζ to be of the form $\sum\zeta_{H,\phi_i}$, where $\phi_i(H)\cong K$ is constant in i

We now proceed to the proofs of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. If $\mathscr{M}_0 = S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K)$, then $\mathscr{M} = S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K)$ as well and we are done. So, assume $(S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K))/\mathscr{M}_0 \neq 0$. Let m be an element of $S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K)$ which is not in $\mathscr{M}_0$. Since $m \notin \mathscr{M}_0$, the definition of $\mathscr{M}_0$ implies that there is a surjection $\phi : L \twoheadrightarrow K_0$ so that $\zeta_{L,\phi} \cdot m \neq 0$, where $K \cong K_0 \leq P$ and $L \leq P$. By (3.1), we can write $\zeta_{L,\phi} \cdot m = s_0 \otimes \bar{f}_{K_0,\psi_0}$ where $s_0 \neq 0$ in S.

As $S \otimes_{\mathbf{F}_p\mathrm{Out}K} \bar{L}(P,K) \neq \mathscr{M}_0$, there exists $K' \cong K$ and $s' \in S$ so that $s' \otimes \bar{f}_{K',\psi} \notin \mathscr{M}_0$. Since $s_0 \neq 0$, Remark 3.4 implies there is an element $\zeta = \sum_{i=1}^{N} \zeta_{L,\phi_i}$ so that $\zeta \cdot m = s' \otimes \bar{f}_{K',\psi}$, where $\phi_i(L) = K'$ for all i. As $s' \otimes \bar{f}_{K',\psi} \notin \mathscr{M}_0$, we must have that $\zeta_{L,\phi_i} \cdot m \notin \mathscr{M}_0$ for some i. So, by Definition 2.2, $m \notin \mathscr{M}_1$. Therefore, $\mathscr{M}_1 = \mathscr{M}_0$ which implies that $\mathscr{M} = \mathscr{M}_0$ $\square$

Proof of Lemma 3.2. Since $\bar{L}(P,K,S) \neq 0$, it is simple, by 5.7 of [1]. So, we have a summand X of BP with vertex K and source S. Let $i : X \to BP$ and $\pi : BP \to X$ satisfy $\pi \circ i = \mathrm{id}_{\{X,X\}}$ so that $e = i \circ \pi$ is the primitive idempotent splitting X from BP. By Lemma 4.1 of [14],

$$\{X,X\} \xrightarrow{\cong} e\{BP,BP\}e$$

via $f \mapsto i \circ f \circ \pi$. Since e is primitive, $e\{BP,BP\}e$ is a local ring and so is $\{X,X\}$. The composition $\{BP,BP\} \xrightarrow{\cong} \widehat{A}(P,P) \twoheadrightarrow \bar{A}(P,P)$ sends e to $\bar{e} \in \bar{A}(P,P)$, the primitive idempotent corresponding to $\bar{L}(P,K,S)$. We get a sequence of ring maps

$$\{X,X\} \xrightarrow{\cong} e\{BP,BP\}e \twoheadrightarrow \bar{e}\bar{A}(P,P)\bar{e} \cong \mathrm{End}_{\bar{A}(P,P)}(\bar{A}(P,P)\bar{e}).$$

As $\bar{A}(P,P)\bar{e}$ is the projective cover of $\bar{L}(P,K,S)$, we get that $\mathrm{End}_{\bar{A}(P,P)}(\bar{A}(P,P)\bar{e})$ maps onto $\mathrm{End}_{\bar{A}(P,P)}(\bar{L}(P,K,S))$ which is a field. Since $\{X,X\}$ is a local ring, we have that

$$\{X,X\}/\mathrm{Rad}\{X,X\} \cong \mathrm{End}_{\bar{A}(P,P)}\bar{L}(P,K,S).$$

Now, let $\bar{e}_0 \in \mathbf{F}_p\mathrm{Out}K$ be a primitive idempotent satisfying $\bar{e}_0 S \neq 0$. We get $\bar{e}_0$ equals the image of e_0 under the composition $\{BK,BK\} \twoheadrightarrow \bar{A}(K,K) \twoheadrightarrow \mathbf{F}_p\mathrm{Out}K$, where $e_0 \in \{BK,BK\}$ is the primitive idempotent splitting X from BK. We have

$$\begin{aligned}\{X,X\} &\xrightarrow{\cong} e_0\{BK,BK\}e_0 \twoheadrightarrow \bar{e}_0\mathbf{F}_p\mathrm{Out}K\bar{e}_0 \\ &\cong \mathrm{End}_{\mathbf{F}_p\mathrm{Out}K}(\bar{e}_0\mathbf{F}_p\mathrm{Out}K) \twoheadrightarrow \mathrm{End}_{\mathbf{F}_p\mathrm{Out}K}(S)\end{aligned}$$

and so $\{X,X\}/\mathrm{Rad}\{X,X\} \cong \mathrm{End}_{\mathbf{F}_p\mathrm{Out}K}(S)$. The result follows $\square$

We finish this section with some technical results from [2] which we will refer to in the following section. Many more such lemmas appear in [2]. The first three lemmas concern computing $\bar{L}(P,K,\mathbf{F}_p)$ when $S = \mathbf{F}_p$ is a source. In this case, $s \otimes \bar{f}_{K',\psi} = s \otimes \bar{f}_{K',\psi'}$ for any isomorphisms $\psi, \psi' : K' \to K$.

Lemma 3.5. *Let* $1 \otimes \bar{f}_{K_i,\psi_i} \in \mathbf{F}_p \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)$ *for* $1 \le i \le n$ *and let* $\phi : P \twoheadrightarrow K_0$ *where* $K_0 \le P$ *and* $K_0 \cong K$. *Then*

$$\zeta_{P,\phi} \cdot \left(\sum_{i=1}^{n} 1 \otimes \bar{f}_{K_i,\psi_i} \right) = N(1 \otimes \bar{f}_{K_0,\psi_0}),$$

where N *is the number of* i *for which* $\ker \phi \cap K_i = 1$ *and* $\psi_0 : K_0 \xrightarrow{\cong} K$.

Proof. Note that 1 represents the only double coset of K_i and P in P. By (2.1), we get $\zeta_{P,\phi}(1 \otimes \bar{f}_{K_i,\psi_i}) = 1 \otimes \bar{f}_{K_0,\psi_0}$ if $\ker \phi \cap K_i = 1$ and 0 otherwise. The result follows □

Lemma 3.6. *Let* $1 \otimes \bar{f}_{K_i,\psi_i} \in \mathbf{F}_p \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)$, $1 \le i \le N$, *and* $\phi : L \twoheadrightarrow K_0$, *where* $K_0 \le P$ *is isomorphic to* K. *Let* n_i *be the number of double coset representatives* $x \in [K_i \backslash P/L]$ *satisfying* $K_i \le {}^xL$ *and* $K_i \cap {}^x\!\ker \phi = 1$. *Then*

$$\zeta_{L,\phi} \cdot \left(\sum_{i=1}^{N} 1 \otimes \bar{f}_{K_i,\psi_i} \right) = \left(\sum_{i=1}^{N} n_i \right) (1 \otimes \bar{f}_{K_0,\psi_0}).$$

Proof. This follows immediately from (2.1) and the comments preceding Lemma 3.5. As in (2.1), we can replace the condition $K_i \le {}^xL$ with $K_i \le \mathrm{Stab}_P(S) \le {}^xL$ □

Lemma 3.7. *Let* $K, L \le P$ *with* $L \ne P$. *Let* $\phi : L \twoheadrightarrow K_0$ *with* K_0 *of the same type as* K. *If* $N_P(L) \le N_P(\ker \phi)$, *then* $\zeta_{L,\phi}$ *annihilates* $\mathbf{F}_p \otimes_{\mathbf{F}_p \mathrm{Out}\, K} \bar{L}(P,K)$. *Similarly, if* $H = L$ *and* $\pi = \phi$, *the element* γ *in* $\mathbf{F}_p OutK$ *in Theorem* 2.3 *associated with* H *and* π *annihilates* $\mathbf{F}_p$.

Proof. Let $K' \le P$, $K' \cong K$. Suppose that $x \in [K' \backslash P/L]$ satisfies $K' \le {}^xL$ and $K' \cap {}^x\!\ker \phi = 1$. Note that this implies that $K'xL = xL$ is a left coset of L in P. Let $g \in N_P(L)$. Then, $K' \le {}^{xg}L$ and ${}^{xg}\!\ker \phi \cap K' = {}^x\!\ker \phi \cap K' = 1$, since $N_P(L) \le N_P(\ker \phi)$. This implies that the number of double coset representatives $y \in [K' \backslash P/L]$ satisfying $K' \le {}^yL$ and $K' \cap {}^y\!\ker \phi = 1$ is a multiple of $[N_P(L) : L]$, which is divisible by p. Lemma 3.6 then implies that $\zeta_{L,\phi}(1 \otimes \bar{f}_{K',\psi}) = 0$. The last statement of the lemma follows by a similar argument (see comments preceding Proposition 2.4) □

We say K is a *retract* of H if there is a split surjection $\phi : H \twoheadrightarrow K$. If $\ker \phi \cap K' = 1$ for all direct summands K' of H isomorphic to K and every such surjection ϕ, we say K is an *absolute retract* of H.

Lemma 3.8. *Let* $K, H < P$ *with* $[P : H] = p$, *and* $H \le N_P(K')$ *for all* $K' \le H$ *such that* $K' \cong K$. *Suppose that, for each* $\pi : H \twoheadrightarrow K$, $\ker \pi \cap K' = 1$ *for all or none of the subgroups* K' *of* H *which are isomorphic to* K *and are direct summands of* H.

Then, a subgroup K' of H with $K' \cong K$ contributes to the multiplicity of the principal dominant summand of BK in BP if and only if K' is a retract of P.

Proof. If K' is a retract of P, then K' obviously contributes to the multiplicity of the principal dominant summand of BK in BP.

Suppose $K' \leq H$ is not a retract of P. For $S = \mathbf{F}_p$, $\mathrm{Stab}_P(S) = N_P(K') \geq H$, and so we need only consider $\pi : H \twoheadrightarrow K'$ when applying Theorem 2.3 with this K'. By Corollary 2.5, K' does not contribute to the multiplicity of the principal dominant summand of BK in BP unless K' is a direct summand of $N_P(K')$. So, assume K' is a summand of $N_P(K')$ and thus also of H. Since H is normal in P any conjugate of K' is also contained in H, and this conjugate will still be a direct summand of H. If $\ker \pi$ intersects all direct summands of H isomorphic to K trivially, we get $\gamma = \sum_{x \in [P/H]} c_{x^{-1}}$. Since $\gamma \mathbf{F}_p = 0$, the result follows in this case. If $\ker \pi$ intersects none of the direct summands of H isomorphic to K trivially, then $\gamma = 0$. This proves the lemma $\square$

4. Examples

The two examples of spaces BP for which we give stable splittings are those for which $P = \Gamma_3 f$ and $P = \Gamma_2 g$, in the notation of Hall and Senior [7]. The first group is a non-split extension of $\mathbf{Z}/4$ by $\mathbf{Z}/8$, and is one of the three nonabelian groups of order 32 for which BP is indecomposable. The second is a semi-direct product of $\mathbf{Z}/8 \times \mathbf{Z}/2$ and $\mathbf{Z}/2$, with all cyclic subgroups of order 8 normal in P and only one central involution. Of the 32 spaces for which we give splittings in [2], these are among the easiest, but the reader will get a flavor of the arguments found there.

In addition to the results of Sections 2 and 3, we require a substantial amount of information about P and its subgroups to achieve our splittings, for which we refer to Hall and Senior [7]. For each group P we include

(i) A lattice diagram of normal subgroups, hereafter referred to as the diagram, or the diagram for P, K, etc. (see below)

(ii) The order structure of P, denoted $\mathrm{OS}(P)$ (to be explained below).

(iii) Information concerning the automorphisms of P.

(iv) Generators and relations determining P.

(v) A table listing the subgroups K of P along with information about these K including some of the above information given for P as well as $N_P(K)$ and usually $KC_P(K)$.

Each box (circles and diamonds will also be referred to as boxes) of a diagram represents one or more normal subgroups of K isomorphic to the group listed in the bottom half of the box, under the line. The group listed in the top half of the box, over the line, is the quotient of K by the normal subgroup listed under the line. If there are parentheses and an exponent around the 'fraction' in a box, the exponent denotes the number of subgroups represented by the box. Nonabelian groups will be labelled using the notation of [7]. Abelian groups will be labelled by their invariants. For example, $1^3 = (\mathbf{Z}/2)^3$ while $211 = \mathbf{Z}/4 \times (\mathbf{Z}/2)^2$.

Bold boxes denote characteristic subgroups. Circles denote a member of the ascending central series. Diamond shaped boxes represent the derived series.

A number directly below a box tells us how many of the subgroups in the box below (and along the same line) are contained in each of the subgroups above. Similarly, a number directly above a box tells us how many of the groups above contain each of the groups below.

The order structure of P gives the number of elements of order 2, 4, 8, etc. For example, $\mathrm{OS}(32\Gamma_2 f) = (7, 24)$ which means that $\Gamma_2 f$ has seven elements of order 2 and 24 of order 4 (in addition to the identity).

The order of Aut$P$ is given as a product $t_1 \cdot t_2$. Here, t_1 is the number of automorphisms of P which act as the identity on $P/\Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of P. Also, t_1 is the order of the kernel of the map $\mathrm{Aut}P \to \mathrm{Aut}(P/\Phi(P))$. The order of the image, which we denote $\mathscr{G}$ or $\mathscr{G}(\mathrm{P})$, is t_2 and this will also be the order of the image of $\Phi_*: \mathrm{Out}P \to \mathrm{Out}(P/\Phi(P))$. Φ_* extends to a ring homomorphism $\mathbf{F}_p\mathrm{Out}P \to \mathbf{F}_p\mathrm{Out}(P/\Phi(P))$ which has image $\mathbf{F}_p\mathscr{G}$. The simple $\mathbf{F}_p\mathscr{G}$-modules will be the same as the simple $\mathbf{F}_p\mathrm{Out}P$-modules under extension of scalars by this ring homomorphism (see [6, p. 175]).

Generators and relations will be given as in [7]. The square of each generator α_i will be given as well as commutators $[\alpha_i, \alpha_j] = \alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1}$ for $i < j$. If such a commutator is equal to the identity it will not be listed.

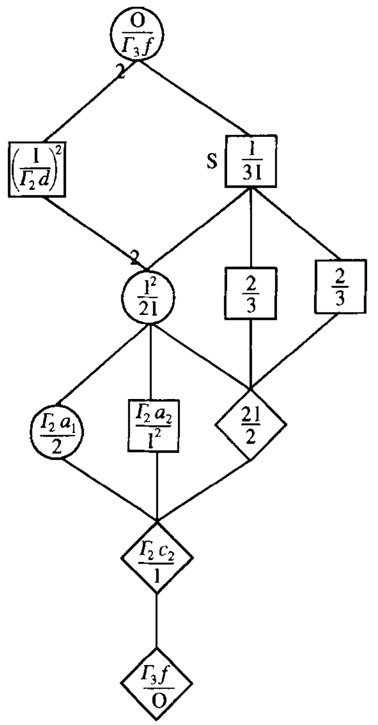
The information in the tables mentioned above (see Table 1) can be inferred from [7] directly or through elementary group-theoretic arguments. The first column gives the subgroups of P. The second column tells how many copies of a subgroup are designated by a particular row of the table. We have only bothered to determine $KC_P(K)$ when it was trivial to do so or necessary to our calculations.

We will not always list all the subgroups isomorphic to a given group in the same row. The multiplicity number will only apply to the groups designated in that row. Also, note that the rows in the table will not necessarily correspond to the boxes in the lattice diagrams.

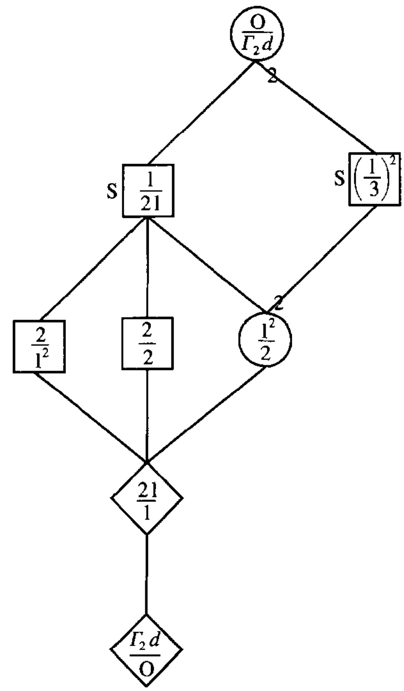
Finally, we will often put capital letters $A, B, \ldots$, or $A_i, B_i, \ldots$, at the very left of some of the rows to denote the group or groups in that row. This is just a notational convenience and such letters will be used to refer to these groups in the following arguments.

Indecomposable summands will be labelled by vertex and source, unless there is some other conventional notation. For example, the principal dominant summand of BQ_8 will be denoted $X_{Q_8,\mathbf{F}_2}$ while the dominant summand of $(\mathbf{Z}/2)^n$ corresponding to the Steinberg module will be denoted $L(n)$, as is customary. Usually, there are at most two simple $\mathbf{F}_2\mathrm{Out}K$-modules. If there are two, then $X_{K,S}$ denotes the nonprincipal dominant summand, S denoting the nontrivial simple $\mathbf{F}_2\mathrm{Out}K$-module. If K is indecomposable, we will write $\overline{BK}$ for the unique nontrivial summand of BK. These conventions hold in the tables of Section 5 as well.

$P = 32\,\Gamma_3 f$



$16\Gamma_2 d$



Theorem 4.1. *For $P = 32\Gamma_3 f$, BP is indecomposable.*

Proof. From the diagrams for $16\Gamma_2 d$ and P, and Table 1, we see that only those subgroups K listed in the last row of Table 1 are direct summands of both $KC_P(K)$ and $N_P(K)$. These K are the two direct summands of B isomorphic to $\mathbf{Z}/2$. By Proposition 2.4 and Corollary 2.5, these subgroups are the only possible contributors.

We claim $\mathbf{Z}/2$ is not a vertex of BP. All the involutions of P are contained in $\Phi(P)$ and so there is no split surjection $P \twoheadrightarrow \mathbf{Z}/2$. So, we need only consider $H = B$ when applying Theorem 2.3. As K is an absolute retract of B, Lemma 3.8 implies that $\mathbf{Z}/2$ is not a vertex of BP. This proves the theorem $\square$

Theorem 4.2. *For $P = 32\Gamma_2 g$, we have*

$$BP \simeq B1 \vee 2\overline{B\mathbf{Z}/2} \vee X_{P,\mathbf{F}_2} \vee 2X_{P,S},$$

where S is the nontrivial simple $\mathbf{F}_p\mathrm{Out}P$-module.

Proof. By Proposition 2.4 and Corollary 2.5, we need only consider the subgroups in the last row of Table 2 as possible contributors, since P is not a direct product in

Table 1
$P = \{\alpha_i, 1 \le i \le 5 \mid \alpha_1^2 = 1, \alpha_2^2 = \alpha_5^2 = \alpha_1, \alpha_3^2 = \alpha_2^3, \alpha_4^2 = \alpha_5, [\alpha_2,\alpha_4] = \alpha_1, [\alpha_3,\alpha_4] = \alpha_2, \} |\mathrm{Aut}P| = 2^5 \cdot 2, \mathrm{OS}(P) = (3,4,24)$

	Sbgps. K	Mult.	OS(K)	$\lvert$Aut(K)$\rvert$	$N_P(K)$	$KC_P(K)$
A_i	$\Gamma_2 d$	2	(3,4,8)	$8 \cdot 2$	P	K
B	$\mathbf{Z}/8 \times \mathbf{Z}/2$	1	(3,4,8)	$8 \cdot 2$	P	K
$\Phi(P)$	$\mathbf{Z}/4 \times \mathbf{Z}/2$	1	(3,4)	$4 \cdot 2$	P	K
	$\mathbf{Z}/8$	2	(1,2,4)	$4 \cdot 1$	P	B
	$\mathbf{Z}/8$	4	(1,2,4)	$4 \cdot 1$	A_i	K
	$(\mathbf{Z}/2)^2$	1	(3)	$1 \cdot 6$	P	B
	$\mathbf{Z}/4$	2	(1,2)	$2 \cdot 1$	P	—
	$\mathbf{Z}/2$	1	(1)	1	P	P
	$\mathbf{Z}/2$	2	(1)	1	B	B

Table 2
$P = \{\alpha_i, 1 \le i \le 4 \mid \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1, \alpha_4^4 = \alpha_1, [\alpha_2,\alpha_3] = \alpha_1\}, |\mathrm{Aut}P| = 2^4 \cdot 6, \mathrm{OS}(P) = (7,8,16)$

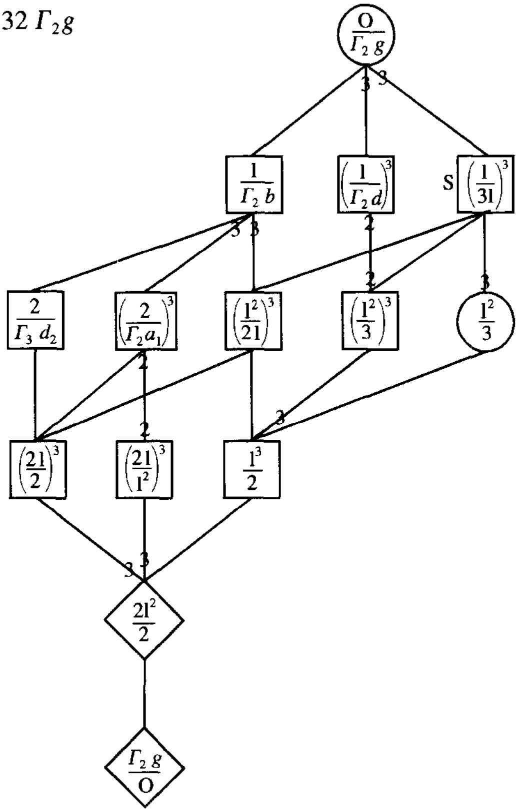
	Sbgps. K	Mult.	OS(K)	$\lvert$Aut(K)$\rvert$	$N_P(K)$	$KC_P(K)$
A	$\Gamma_2 b$	1	(7,8)	$2^3 \cdot 6$	P	P
B_i	$\Gamma_2 d$	3	(3,4,8)	$2^3 \cdot 2$	P	P
C_i	$\mathbf{Z}/8 \times \mathbf{Z}/2$	3	(3,4,8)	$2^3 \cdot 2$	P	K
	$\Gamma_2 a_2$	1	(1,6)	$4 \cdot 6$	P	P
	$\Gamma_2 a_1$	3	(5,2)	$4 \cdot 2$	P	P
	$\mathbf{Z}/4 \times \mathbf{Z}/2$	3	(3,4)	$4 \cdot 2$	P	C_i
	$Z(P)\mathbf{Z}/8$	1	(1,2,4)	$4 \cdot 1$	P	P
	$\mathbf{Z}/8$	3	(1,2,4)	$4 \cdot 1$	P	C_i
	$\mathbf{Z}/4$	1	(1,2)	$2 \cdot 1$	P	P
	$\mathbf{Z}/4$	3	(1,2)	$2 \cdot 1$	P	C_i
	$(\mathbf{Z}/2)^2$	3	(3)	$1 \cdot 6$	P	C_i
P'	$\mathbf{Z}/2$	1	(1)	1	P	P
	$\mathbf{Z}/2$	6	(1)	1	C_i	C_i

a nontrivial way. We claim that $\mathbf{Z}/2$ is a vertex of BP and $\overline{B\mathbf{Z}/2}$ is a summand of multiplicity two in BP.

From the diagram for P and the table, we see that there are three subgroups of P isomorphic to $(\mathbf{Z}/2)^2$ and all contain P', which is the commutator subgroup and the one central involution of P. So, these three subgroups will each contain two of the subgroups K in the last row of the table and each of these pairs will form a conjugacy class, whose representatives we will denote K_1, K_2, and K_3. By Lemma 2.1, we have that $\mathbf{F}_2 \otimes \mathbf{F}_{2\mathrm{Out}K}\bar{L}(P,K)$ is three-dimensional. From the order structures of the groups of index two in P and the diagram for P, we get that A contains all the subgroups K and each B_i and C_i contain a unique conjugacy class of such K. We number B_i and C_i so that K_i represents the conjugacy class contained in $B_i \cap C_i$, for $1 \le i \le 3$.

We have that a homomorphism $\pi : P \twoheadrightarrow K$ with kernel B_i or C_i will satisfy $\ker \pi \cap K_j = 1$ if and only if $i \ne j$. By Theorem 2.3, we get that each K_i contributes to the multiplicity

$P = 32\,\Gamma_2 g$



of $\overline{B\mathbf{Z}/2}$ in BP, and so $\mathbf{Z}/2$ is a vertex. When computing $\mathscr{M}$ via Definition 2.2, we need only consider $L = P$ or $L = C_j$, since these are the only subgroups of P which contain any of the groups $\mathrm{Stab}_P(\mathbf{F}_2)$. If $\phi : P \twoheadrightarrow K_1$ with kernel B_j or C_j, we get

$$\zeta_{P,\phi}(1 \otimes \bar{f}_{K_i,\psi_i}) = \begin{cases} 1 \otimes \bar{f}_{K_1,\psi_1} & \text{if } i \neq j; \\ 0 & \text{if } i = j, \end{cases}$$

by Lemma 3.5. From this, one easily sees that $\mathscr{M} \leq \langle \sum_{i=1}^{3} 1 \otimes \bar{f}_{K_i,\psi_i} \rangle$ and $\sum_{i=1}^{3} 1 \otimes \bar{f}_{K_i,\psi_i}$ is annihilated by any $\zeta_{P,\phi}$ as above. If $\ker\phi = A$, then (2.1) gives $\zeta_{P,\phi} \cdot \bar{f}_{K_i,\psi_i} = 0$ since the condition $K_i \cap {}^x\ker\phi = 1$ cannot be satisfied.

Next, consider $L = C_i$. Since $\ker\phi \cong \mathbf{Z}/8$, we see from Table 2 that $\ker\phi$ is normal in P. By Lemma 3.7, $\zeta_{C_i,\phi}$ annihilates $\mathbf{F}_2 \otimes_{\mathbf{F}_2}$ It follows that

$$\mathscr{M} = \langle \bar{f}_{K_1,\psi_1} + \bar{f}_{K_2,\psi_2} + \bar{f}_{K_3,\psi_3} \rangle.$$

Since the endomorphism ring of $\mathbf{F}_2$ is $\mathbf{F}_2$, Lemma 3.2 implies that $\bar{L}(P,K,\mathbf{F}_2)$ is two-dimensional over its endomorphism ring.

To finish the proof, we need only consider the dominant summands of BP. From the diagram for P, we see that $P/\Phi(P) \cong (\mathbf{Z}/2)^2$, and from [7], we have that $\mathscr{G}(\mathrm{P})$ is of order six. So, $\Phi_* : \mathrm{Out}P \to \mathrm{Out}(P/\Phi(P))$ is surjective. There are two absolutely simple $\mathbf{F}_2\mathrm{Out}(P/\Phi(P))$-modules, $\mathbf{F}_2$ and the Steinberg module St, and $\mathbf{F}_2$ is the endomorphism ring for both. So, there are two simple $\mathbf{F}_p\mathrm{Out}P$-modules, $\mathbf{F}_2$ and S, and the latter is two-dimensional over its endomorphism ring. The corresponding summands appear with multiplicity one and two respectively and the theorem follows. □

5. Tables

In the table of splittings, we have omitted the $B1$ summand. Also, we have the following alternative descriptions of some of the spectra which appear.

$\overline{B\mathbf{Z}/2} \simeq L(1)$	$\overline{BA_4} \simeq X_{(\mathbf{Z}/2)^2,\mathbf{F}_2}$	$L(2) \simeq X_{(\mathbf{Z}/2)^2,St}$
$X_{D_8,\mathbf{F}_2} \simeq \overline{BPSL_2(\mathbf{F}_7)}$	$X_{Q_8,\mathbf{F}_2} \simeq \overline{BSL_2(\mathbf{F}_5)}$	$X_{Q_8,S} \simeq \sum^{-1}(BS^3/BN)$
$L(3) \simeq X_{(\mathbf{Z}/2)^3,St}$	$L(2) \vee L(3) \simeq M(3)$	$\overline{BP} \simeq X_{P,\mathbf{F}_2}$
$X_{\Gamma_8 a_1,\mathbf{F}_2} \simeq \overline{BPSL_2(\mathbf{F}_{31})}$	$X_{\Gamma_8 a_2,\mathbf{F}_2} \simeq \overline{BSL_3(\mathbf{F}_7)}$	$X_{\Gamma_8 a_3,\mathbf{F}_2} \simeq \overline{BSL_3(\mathbf{F}_{17})}$

Here, $L(n) \simeq \sum^{-n} Sp^{2^n}S^0/Sp^{2^{n-1}}S^0$ and the N which appears in $X_{Q_8,S}$ is the normalizer of the maximal torus of S^3 (see [13]). In addition, $B(SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3))$ is stably equivalent to $B1 \vee \overline{BA_4} \vee X_{\Gamma_5 a_1,\mathbf{F}_2}$ and this splitting is complete (compare with [4, p. 493]). The equivalences $X_{\Gamma_8 a_1,\mathbf{F}_2} \simeq \overline{BPSL_2(\mathbf{F}_{31})}$ and $X_{\Gamma_8 a_3,\mathbf{F}_2} \simeq \overline{BSL_3(\mathbf{F}_{17})}$ can be found in [13], and the equivalence $X_{\Gamma_8 a_2,\mathbf{F}_2} \simeq \overline{BSL_3(\mathbf{F}_7)}$ in [11, Ch. 5]. The simple $\mathbf{F}_2\mathrm{Out}P$-module S which appears in the notation $X_{P,S}$ is two-dimensional for all P. It is often the pullback of the two-dimensional Steinberg module. The modules S_1 and S_2 appearing in the entries for $P = \Gamma_5 a_1$ and $P = \Gamma_5 a_2$ are four-dimensional. For more details, see [2, pp. 153, 156]. The modules S and $S^\#$ appearing in the summands $X_{(\mathbb{Z}/2)^3,S}$ and $X_{(\mathbb{Z}/2)^3,S^\#}$ are the standard three-dimensional representation of $(\mathbb{Z}/2)^3$ and its contragredient. We have the following more well-known names for some of the groups:

$$\Gamma_2 k \cong QD_{32}, \quad \Gamma_3 e \cong \mathbf{Z}/4 \wr \mathbf{Z}/2, \quad \Gamma_5 a_1 \cong E(4)^+, \quad \Gamma_5 a_2 \cong E(4)^-,$$

$$\Gamma_8 a_1 \cong D_{32}, \quad \Gamma_8 a_2 \cong SD_{32}, \quad \Gamma_8 a_3 \cong Q_{32}.$$

In the table of series, the series appear in family order, except where the series given is not that of an indecomposable summand. These appear at the end of the table. If a series is too long to fit in the table, it appears after the table. The power of t given at the left of the leading coefficients indicates that all lower powers of t have coefficient 0. If dots $(\cdots)$ appear after the coefficients, one is to assume that the, hopefully obvious, pattern continues. In particular,

$$(1\ 2\ 1\ 2\ 3 \cdots) = (1\ 2\ 1\ 2\ 3\ 2\ 3\ 4\ 3\ 4\ 5 \cdots)$$

$$(1\ 1\ 0\ 0 \cdots) \quad = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0 \cdots)$$

Finally, $L_2(t), L_3(t)$, and $M_3(t)$ are the Poincaré series for $L(2)$, $L(3)$ and $M(3) = L(3) \vee L(2)$ respectively. The first two of these appear in the table and $M_3(t) = L_3(t) + L_2(t)$.

Table of splittings for groups of order 32

P	The complete splitting of $\overline{BP}$
$\Gamma_2 f$	$\overline{B\mathbf{Z}/2} \vee \overline{B\mathbf{Z}/4} \vee X_{\mathbf{Z}/4\times(\mathbf{Z}/2)^2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_2 g$	$2\overline{B\mathbf{Z}/2} \vee X_{P,\mathbf{F}_2} \vee 2X_{P,S}$
$\Gamma_2 h$	$2\overline{B\mathbf{Z}/4} \vee X_{P,\mathbf{F}_2} \vee 2X_{P,S}$
$\Gamma_2 i$	$\overline{B\mathbf{Z}/4} \vee X_{P,\mathbf{F}_2}$
$\Gamma_2 j_1$	$\overline{B\mathbf{Z}/2} \vee L(2) \vee \overline{B\mathbf{Z}/8} \vee X_{\mathbf{Z}/2\times\mathbf{Z}/4,\mathbf{F}_2} \vee X_{\mathbf{Z}/4\times(\mathbf{Z}/2)^2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_2 j_2$	$\overline{B\mathbf{Z}/8} \vee X_{P,\mathbf{F}_2}$
$\Gamma_2 k$	$\overline{B\mathbf{Z}/2} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 b$	$3\overline{B\mathbf{Z}/2} \vee 2X_{16\Gamma_2 b,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 c_1$	$\overline{B\mathbf{Z}/2} \vee \overline{B\mathbf{Z}/4} \vee \overline{BA_4} \vee 4L(2) \vee 4L(3) \vee X_{(\mathbf{Z}/2)^3,S} \vee X_{(\mathbf{Z}/2)^3,S^\#} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 c_2$	$\overline{B\mathbf{Z}/4} \vee X_{Q_8,S} \vee X_{Q_8\times\mathbf{Z}/2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 d_1$	$\overline{B\mathbf{Z}/4} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 d_2$	$\overline{B\mathbf{Z}/4} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 e$	$\overline{B\mathbf{Z}/2} \vee \overline{B\mathbf{Z}/4} \vee X_{16\Gamma_2 b,S} \vee X_{(\mathbf{Z}/4)^2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_3 f$	$\overline{BP}$
$\Gamma_4 a_2$	$3\overline{B\mathbf{Z}/2} \vee 8L(2) \vee 8L(3) \vee 2X_{D_8,\mathbf{F}_2} \vee X_{P,\mathbf{F}_2} \vee 2X_{P,S}$
$\Gamma_4 a_3$	$X_{Q_8,\mathbf{F}_2} \vee 3X_{Q_8,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_4 b_1$	$3\overline{B\mathbf{Z}/2} \vee 5L(2) \vee 4L(3) \vee 2X_{D_8,\mathbf{F}_2} \vee X_{\mathbf{Z}/2\times\mathbf{Z}/4,\mathbf{F}_2} \vee X_{\mathbf{Z}/4\times(\mathbf{Z}/2)^2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_4 b_2$	$\overline{B\mathbf{Z}/2} \vee L(2) \vee X_{\mathbf{Z}/2\times\mathbf{Z}/4,\mathbf{F}_2} \vee X_{Q_8,\mathbf{F}_2} \vee 2X_{Q_8,S} \vee X_{\mathbf{Z}/4\times(\mathbf{Z}/2)^2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_4 c_1$	$2\overline{B\mathbf{Z}/2} \vee 2L(2) \vee 2L(3) \vee X_{\mathbf{Z}/4\times(\mathbf{Z}/2)^2,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_4 c_2$	$2\overline{B\mathbf{Z}/2} \vee 4L(2) \vee 4L(3) \vee X_{P,\mathbf{F}_2}$
$\Gamma_4 c_3$	$\overline{BP}$
$\Gamma_4 d$	$\overline{B\mathbf{Z}/2} \vee 2L(2) \vee 2L(3) \vee X_{P,\mathbf{F}_2} \vee X_{P,S}$
$\Gamma_5 a_1$	$4\overline{B\mathbf{Z}/2} \vee 5\overline{BA_4} \vee 12L(2) \vee 12L(3) \vee X_{P,\mathbf{F}_2} \vee 4X_{P,S_1} \vee 4X_{P,S_2}$
$\Gamma_5 a_2$	$4\overline{B\mathbf{Z}/2} \vee X_{P,\mathbf{F}_2} \vee 4X_{P,S_1} \vee 4X_{P,S_2}$
$\Gamma_6 a_1$	$3\overline{B\mathbf{Z}/2} \vee \overline{BA_4} \vee 4L(2) \vee 4L(3) \vee X_{(\mathbf{Z}/2)^3,S} \vee X_{(\mathbf{Z}/2)^3,S^\#} \vee X_{16\Gamma_2 b,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_6 a_2$	$2\overline{B\mathbf{Z}/2} \vee X_{Q_8,S} \vee X_{Q_8\times\mathbf{Z}/2,S} \vee X_{16\Gamma_2 b,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_7 a_1$	$\overline{B\mathbf{Z}/2} \vee \overline{B\mathbf{Z}/4} \vee 4L(2) \vee 4L(3) \vee X_{P,\mathbf{F}_2}$
$\Gamma_7 a_2$	$\overline{B\mathbf{Z}/2} \vee 4L(2) \vee 4L(3) \vee X_{P,\mathbf{F}_2}$
$\Gamma_7 a_3$	$\overline{BP}$
$\Gamma_8 a_1$	$2\overline{B\mathbf{Z}/2} \vee 2L(2) \vee X_{P,\mathbf{F}_2}$
$\Gamma_8 a_2$	$\overline{B\mathbf{Z}/2} \vee L(2) \vee X_{Q_8,S} \vee X_{P,\mathbf{F}_2}$
$\Gamma_8 a_3$	$2X_{Q_8,S} \vee X_{P,\mathbf{F}_2}$

Table of Poincaré series

Spec.	P – series	Lead.coef.
$\overline{B(\mathbf{Z}/2^n)}$	$t/(1-t)$	$t(1\ 1\ 1\ 1\cdots)$
$L(2)$	$t^4/(1-t)(1-t^3)$	$t^4(1\ 1\ 1\ 2\ 2\ 2\cdots)$
$\overline{BA_4}$	$(t^2+t^3-t^4)/(1-t)(1-t^3)$	$t^2(1\ 2\ 1\ 2\ 3\cdots)$
$X_{D_8,\mathbf{F}_2}$	$(t^2+t^3-t^4)/(1-t)(1-t^3)$	$t^2(1\ 2\ 1\ 2\ 3\cdots)$
$X_{Q_8,\mathbf{F}_2}$	$(t^3+t^4)/(1-t^4)$	$t^3(1\ 1\ 0\ 0\cdots)$
$X_{Q_8,S}$	$(t+t^2)/(1-t^4)$	$t(1\ 1\ 0\ 0\cdots)$
$X_{\mathbf{Z}/2\times\mathbf{Z}/4,\mathbf{F}_2}$	$t^2/(1-t)^2$	$t^2(1\ 2\ 3\ 4\ \cdots)$
$L(3)$	$\dfrac{t^{11}+t^{12}+t^{13}+2t^{14}+t^{15}+t^{16}+t^{17}}{(1-t^4)(1-t^6)(1-t^7)}$	$t^{11}(1\ 1\ 1\ 2\ 2\ 2\cdots)$
$X_{(\mathbf{Z}/2)^3,S}$	see $P_1(t)$ below	$t^5(1\ 2\ 2\ 4\ 5\ 5\ 6)$
$X_{(\mathbf{Z}/2)^3,S^\#}$	see $P_2(t)$ below	$t^7(1\ 1\ 2\ 4\ 3\ 4\ 6)$
$X_{\mathbf{Z}/4\times(\mathbf{Z}/2)^2,S}$	$t^5/(1-t)^2(1-t^3)$	$t^5(1\ 2\ 3\ 5\ 7\ 9)$
$X_{(\mathbf{Z}/4)^2,S}$	$t^4/(1-t)(1-t^3)$	$t^4(1\ 1\ 1\ 2\ 2\ 2\cdots)$
$X_{Q_8\times\mathbf{Z}/2,S}$	$(t^2+t^3)/(1-t)(1-t^4)$	$t^2(1\ 2\ 2\ 2\ 3\ 4)$
$X_{16\Gamma_2 b,S}$	$(t^2+t^5)/(1-t)(1-t^4)$	$t^2(1\ 1\ 1\ 2\ 3)$
$X_{\Gamma_2 f,\mathbf{F}_2}$	$\dfrac{(t+t^2+t^3+t^4-t^5-2t^7)}{(1-t)^3(1+t^2)(1+t+t^2)}$	$t(1\ 3\ 5\ 8\ 11\ 14\ 17)$
$X_{\Gamma_2 i,\mathbf{F}_2}$	$t/(1-t)^2$	$t(1\ 2\ 3\ 4\cdots)$
$X_{\Gamma_2 j_1,\mathbf{F}_2}$	$\dfrac{t^2+t^3-t^5}{(1-t)^2(1+t)(1-t^3)}$	$t^2(1\ 2\ 3\ 4\ 6\ 7\ 9)$
$X_{\Gamma_2 j_2,\mathbf{F}_2}$	$t/(1-t)^2$	$t(1\ 2\ 3\ 4\cdots)$
$X_{\Gamma_2 k,\mathbf{F}_2}$	$\dfrac{t-t^2+t^3}{(1-t)^2(1+t^2)}$	$t(1\ 1\ 1\ 2\ 3\ 3\ 2\ 2)$
$X_{\Gamma_3 b,\mathbf{F}_2}$	$\dfrac{t^3-t^5}{(1-t)^2(1-t^4)}$	$t^3(1\ 2\ 2\ 2\ 4\ 4\ 4)$
$X_{\Gamma_3 c_1,\mathbf{F}_2}$	see $P_3(t)$ below	$t^2(1\ 2\ 2\ 3\ 5\ 5\ 7)$
$X_{\Gamma_3 c_2,\mathbf{F}_2}$	$\dfrac{t^3+t^4}{(1-t)(1-t^4)}$	$t^3(1\ 2\ 2\ 2\ 3\ 4\ 4)$
$X_{\Gamma_3 d_1,\mathbf{F}_2}$	$t/(1-t)^2$	$t(1\ 2\ 3\ 4\cdots)$
$X_{\Gamma_3 d_2,\mathbf{F}_2}$	$t/(1-t)^2$	$t(1\ 2\ 3\ 4\cdots)$
$X_{\Gamma_3 e,\mathbf{F}_2}$	$\dfrac{t^3+t^4+t^5-t^7}{(1-t^3)(1-t^4)}$	$t^3(1\ 1\ 1\ 1\ 1\ 2\ 2)$
$X_{\Gamma_3 f,\mathbf{F}_2}$	$\dfrac{2t-2t^2+2t^3-t^4}{(1-t)^2(1+t^2)}$	$t(2\ 2\ 2\ 3\ 4\ 4\ 4)$
$X_{\Gamma_4 a_3,\mathbf{F}_2}$	$\dfrac{t^3+2t^2}{(1-t)^2(1+t^2)}$	$t^2(2\ 5\ 6\ 6\ 8\ 11\ 14)$
$X_{\Gamma_4 b_1,\mathbf{F}_2}$	$\dfrac{t^3+4t^4-3t^5}{(1-t)^2(1-t^3)}-4M_3(t)$	$t^3(1\ 2\ 4\ 7\ 10\ 14\ 18)$

Table of Poincaré series (continued)

Spec.	P − series	Lead.coef.
$X_{\Gamma_4 b_2,\mathbf{F}_2}$	$\dfrac{t^2-t^4-t^6-t^7}{(1-t)^3(1-t^3)(1+t^2)}$	$t^2(1\ 3\ 4\ 5\ 7\ 9\ 11)$
$X_{\Gamma_4 c_1,\mathbf{F}_2}$	see $P_4(t)$ below	$t(1\ 3\ 6\ 8\ 12\ 16\ 19)$
$X_{\Gamma_4 c_2,\mathbf{F}_2}$	$\dfrac{t+t^4-t^5}{(1-t)^3(1+t^2)}-4M_3(t)$	$t(1\ 3\ 5\ 4\ 8\ 12\ 16)$
$X_{\Gamma_4 c_3,\mathbf{F}_2}$	$\dfrac{3t-t^2+2t^3-t^4}{(1-t)^2(1+t^2)}$	$t(3\ 4\ 4\ 6\ 9\ 9\ 8)$
$X_{\Gamma_4 d,\mathbf{F}_2}$	same as $\overline{BA_4}$	
$X_{\Gamma_4 d,S}$	see $P_5(t)$ below	$t(2\ 2\ 2\ 4\ 6\ 6\ 7)$
$X_{\Gamma_5 a_1,\mathbf{F}_2}$	$\dfrac{t^3+t^4+t^6+t^7}{(1-t^2)(1-t^3)(1-t^4)}$	$t^3(1\ 1\ 1\ 3\ 4\ 4\ 6)$
$X_{\Gamma_5 a_2,\mathbf{F}_2}$	$\dfrac{t^2+3t^3+4t^4+3t^5+2t^8-t^{10}}{(1-t^2)(1-t^8)}$	$t^2(1\ 3\ 5\ 6\ 5\ 6\ 7)$
$X_{\Gamma_5 a_2,S_1}$	$\dfrac{t^6+t^9}{(1-t)(1-t^8)}$	$t^6(1\ 1\ 1\ 2\ 2\ 2\ 2)$
$X_{\Gamma_5 a_2,S_2}$	$\dfrac{t^2+t^3+t^4+t^5}{(1-t)(1-t^8)}$	$t^2(1\ 2\ 3\ 4\ 4\ 4\ 4)$
$X_{\Gamma_6 a_1,\mathbf{F}_2}$	see $P_6(t)$ below	$t^3(1\ 1\ 2\ 3\ 3\ 5\ 7)$
$X_{\Gamma_6 a_2,\mathbf{F}_2}$	$\dfrac{t^3-t^5+t^7+t^8}{(1-t)(1-t^8)}$	$t^3(1\ 1\ 0\ 0\ 1\ 2\ 2)$
$X_{\Gamma_7 a_1,\mathbf{F}_2}$	$\dfrac{2t^2-t^4}{(1-t)^3(1+t)}-4M_3(t)$	$t^2(2\ 4\ 3\ 6\ 10\ 14\ 19)$
$X_{\Gamma_7 a_2,\mathbf{F}_2}$	$\dfrac{t-t^2+2t^3-t^4}{(1-t)^3(1+t^2)}-4P_3(t)$	$t(1\ 2\ 4\ 3\ 6\ 9\ 9)$
$X_{\Gamma_7 a_3,\mathbf{F}_2}$	$\dfrac{1+t^2+t^5}{(1-t)^2(1+t^2)(1+t^4)}-1$	$t(2\ 3\ 4\ 4\ 5\ 6\ 6)$
$X_{\Gamma_8 a_1,\mathbf{F}_2}$	$\dfrac{t^2+t^3-t^4}{(1-t)(1-t^3)}$	$t^2(1\ 2\ 1\ 2\ 3\cdots)$
$X_{\Gamma_8 a_2,\mathbf{F}_2}$	$\dfrac{t^3+t^4+t^5-t^7}{(1-t^3)(1-t^4)}$	$t^3(1\ 1\ 1\ 1\ 1\ 2)$
$X_{\Gamma_8 a_3,\mathbf{F}_2}$	$(t^3+t^4)/(1-t^4)$	$t^3(1\ 1\ 0\ 0\cdots)$
$X_{\Gamma_2 g,\mathbf{F}_2}\vee 2X_{\Gamma_2 g,S}$	$\dfrac{(t+t^2+t^4)}{(1-t)^2(1+t^2)}$	$t(1\ 3\ 4\ 5\ 7\ 9\ 10)$
$X_{\Gamma_2 h,\mathbf{F}_2}\vee 2X_{\Gamma_2 h,S}$	$\dfrac{(3t^2+2t^3-t^4-t^5)}{(1-t)^2(1+t^2)}$	$t^2(3\ 5\ 10\ 13\ 20\ 24\ 33)$
$X_{\Gamma_4 a_2,\mathbf{F}_2}\vee 2X_{\Gamma_4 a_2,S}$	$\dfrac{t^3+7t^4-5t^5+t^6-2t^8+t^9}{(1-t)^2(1-t^3)}-8P_3(t)$	$t^3(1\ 1\ 4\ 8\ 11\ 17)$
$X_{\Gamma_5 a_1,S_1}\vee X_{\Gamma_5 a_1,S_2}$	$\dfrac{t^4+t^5+t^6+t^7}{(1-t)(1-t^3)(1-t^4)}-L_2(t)-3L_3(t)$	$t^5(1\ 2\ 3\ 3\ 3\ 3\ 2)$

$$P_1(t) = \frac{t^5 + 2t^6 + 2t^7 + 4t^8 + 4t^9 + 3t^{10} + 3t^{11} + t^{12} - t^{14} - t^{15} - t^{16} - t^{17}}{(1-t^4)(1-t^6)(1-t^7)}$$

$$P_2(t) = \frac{t^7 + t^8 + 2t^9 + 4t^{10} + 2t^{11} + 3t^{12} + 3t^{13} + t^{15} - t^{17}}{(1-t^4)(1-t^6)(1-t^7)}$$

$$P_3(t) = \frac{2t^2 - t^4}{(1-t)^3(1+t)} - \frac{t^2 + t^3 - t^4}{(1-t)(1-t^3)} - 4M_3(t) - P_1(t) - P_2(t)$$

$$P_4(t) = \frac{1 + t^2 + t^3}{(1-t)^3(1+t)(1+t^2)} - \frac{1 - t^2 - t^3 + 2t^5}{(1-t)^2(1-t^3)} - 2M_3(t)$$

$$P_5(t) = \frac{2t - 2t^2 + 4t^3 - 2t^4 + 2t^5 - t^7 + t^8 - t^9}{(1-t)^2(1+t^2)^2(1-t^3)}$$

$$P_6(t) = \frac{t^3 + 4t^4 + 3t^5 + 5t^6 + 2t^7 - t^8 - 2t^{10}}{(1-t)(1+t^2)(1-t^3)(1-t^4)} - 4M_3(t) - P_1(t) - P_2(t)$$

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